Journal of Engineering Mathematics, Vol. 5, No. 1, January 1971
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

Separation and the Compressible Boundary Layer

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(Received June 9, 1970)

SUMMARY

The behavior of the compressible boundary layer equations close to a point of zero skin-friction is studied using a perturbation technique of Kaplun. The behavior of the skin friction is reduced to a study of a nonlinear integral equation with an Abel kernel. For a cold wall this yields the singular behavior described by Buckmaster [2] from a different point of view. For a hot wall the behavior is apparently regular when the heat transfer is non zero, in agreement with Stewartson [11]. There is no evidence that the boundary layer breaks down anywhere than at a point of zero skin friction and the implications of this for the similarity solutions of Cohen and Reshotko are discussed.

1. Introduction

One of the most important problems in boundary-layer theory is that of separation. For many years it was believed that integration of the boundary-layer equations must come to an end when the skin-friction vanishes and that the point at which this happens coincides with the point at which the boundary-layer abruptly leaves the surface and induces $O(1)$ disturbances in the exterior inviscid flow. It was not until the work of Goldstein [5] as modified by Stewartson [9] appeared, that convincing mathematical evidence for this point of view was established for the incompressible boundary-layer. They showed that the boundary layer equations break down when the skin-friction vanishes in the sense that the skin-fraction vanishes like $(x_s-x)^{\frac{1}{2}}$ and the displacement thickness has infinite slope at the separation point x_s. This result has an analogy in the behavior of the Falkner–Skan similarity solutions for an exterior velocity x^m when m is negative. Suppose we have a large duct in which the velocity behaves asymptotically like x^m. If $|m|$ is sufficiently small ($m > -0.0904$) the similarity equation has a solution and separation does not occur. If now m is slowly decreased we would expect the limiting skin-friction to fall until eventually separation occurs; the dependence of the skin-friction on m close to this limit is non-analytic. (In fact it was suggested by Hartree [6] that the skin-friction vanishes like $(m+0.09)^{\frac{1}{2}}$ but although the result has apparently been widely accepted his argument is not precise).

In 1962 Stewartson examined the behavior of the compressible boundary layer equations. He came to the conclusion that at a point of vanishing skin friction either the heat transfer vanishes and the skin-friction $\sim (x_s-x)^{\frac{1}{2}}$ or the heat transfer is non-zero and the skin-friction is analytic (we will call this the Stewartson alternative). As a consequence it was speculated (Stewartson, [10]) that for the general compressible boundary-layer, breakdown occurs at some point other than where the skin-friction vanishes. Some evidence for this is afforded by the Falkner–Skan like solutions calculated by Cohen and Reshotko [3]. For a cold wall breakdown occurs for some m before the skin-friction vanishes, whereas for a hot wall breakdown does not occur until after the skin-friction has become negative. Recently however, Buckmaster [2] has shown that the Stewartson alternative is incorrect for a cold wall and the skin-friction vanishes like $(x_s-x)^{\frac{1}{2}} \ln (x_s-x)$. It can not be concluded that breakdown does not occur earlier, but the numerical evidence is against it (Merkin, [8]).

The behavior for a hot wall has not been established and it is the main purpose of this paper to contribute to this problem. The argument given suggests that for a hot-wall the Stewartson alternative is correct. This result is not conclusive but appears to eliminate any other behavior within the framework of the Goldstein–Stewartson analysis. The line of attack is to avoid the trial and error nature of the Goldstein–Stewartson coordinate expansion and instead, formulate the problem as a parameter perturbation. This approach was taken by Kaplun [7] for the incompressible problem and he generated all the features found earlier by Stewartson. Kaplun's method is here applied to the compressible boundary-layer equations.

2. The Perturbation Problem

The boundary layer equations to be examined are those investigated by Stewartson [11] and are appropriate when the Prandtl number is unity, the viscosity is proportional to the absolute temperature and the exterior flow is irrotational and homoenergetic. With y, the distance perpendicular to the wall, scaled by the viscocity the equations are

$$\frac{\partial\Psi}{\partial y}\frac{\partial^2\Psi}{\partial x\partial y}-\frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial y^2}=V\frac{dV}{dx}(1+S)+\frac{\partial^3\Psi}{\partial y^3},$$

$$\frac{\partial\Psi}{\partial y}\frac{\partial S}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial S}{\partial y}=\frac{\partial^2 S}{\partial y^2},\qquad(2.1)$$

$$u=\frac{\partial\Psi}{\partial y},\quad v=-\frac{\partial\Psi}{\partial x}.$$

S is essentially the absolute temperature which is prescribed at the wall where both Ψ and $\partial\Psi/\partial y$ vanish. In addition

$$\frac{\partial\Psi}{\partial y}\to V(x),\quad S\to 0 \text{ as } y\to\infty$$

and S and Ψ are prescribed at some initial station. It is sufficient for the present purpose to assign some constant adverse value to the pressure gradient, namely

$$V\frac{dV}{dx}=-a.\qquad(2.2)$$

Equations 2.1, 2.2 have the exact solution

$$\Psi=\tfrac{1}{2}Dy^2+\tfrac{1}{6}a(1+B)y^3+\tfrac{1}{24}aCy^4,\quad S=B+Cy,\qquad(2.3)$$

which also satisfies the wall boundary-conditions provided the wall temperature is constant. The various coefficients in (2.3) are all constants and this solution is invariant with x and has constant skin-friction D. The essence of Kaplun's method is to choose D to be small, say $D=\varepsilon$, and then perturb the profile (2.3) in a way that ensures diffusion of vorticity away from the wall so that the skin-friction decreases with x, eventually vanishing. At the same time, since in Stewartson's ([11]) formulation the temperature field is inevitably a perturbation about the wall value, we must chose C to be $O(\varepsilon)$. Thus the solution to be perturbed is

$$\Psi_0=\tfrac{1}{6}a(1+B)y^3,\quad S_0=B.\qquad(2.4)$$

At $x=0$ the initial profiles are chosen to be

$$\Psi=\tfrac{1}{6}a(1+B)y^3+\tfrac{1}{2}\varepsilon y^2+\tfrac{1}{24}aA\varepsilon y^4+\varepsilon^2\Psi_2(0,y)+O(\varepsilon^3),\qquad(2.5)$$
$$S=B+A\varepsilon y+\varepsilon^2 S_2(0,y)+O(\varepsilon^3),$$

and a solution to the boundary-layer equations is sought in the form

$$\Psi\sim\Psi_0+\varepsilon\Psi_1+\varepsilon^2\Psi_2+\dots,$$
$$S\sim S_0+\varepsilon S_1+\varepsilon^2 S_2+\dots.$$

At least one of the initial profiles $\Psi_2(0,y)$, $S_2(0,y)$ must be non-zero to avoid the solution (2.3), but their precise choice presents a difficulty. If a singularity is to be avoided at the origin, the initial profiles have to be chosen so that their y-derivatives at the wall satisfy certain compatibility conditions (Goldstein [4]). Uniform flow approaching the leading edge of a flat plate fails to satisfy the necessary conditions for example, and gives rise to the familiar square-root singularity. The difficulty close to separation is that the skin-friction appears in the denominator of many of the compatibility conditions so that in the present problem the incorrect choice of the $O(\varepsilon^2)$ profile can lead to an $O(1)$ failure of the compatability conditions. This would invalidate the choice of (2.4) as the unperturbed flow. An $O(\varepsilon)$ failure would be acceptable and would manifest itself as a singularity in the solution of Ψ_1, S_1, at the origin. It has not been found possible to find any general criterion for the choice in (2.5) but this does not seem to be important provided it can be assumed that some choice will do the job.

Ψ_1 and S_1 satisfy

$$\frac{\partial^3 \Psi_1}{\partial y^3} - \tfrac{1}{2}a(B+1)y^2 \frac{\partial^2 \Psi_1}{dx\,dy} + a(B+1)y\frac{\partial \Psi_1}{\partial x} = aS_1\,, \tag{2.6}$$

$$\frac{\partial^2 S_1}{\partial y^2} - \tfrac{1}{2}a(B+1)y^2\frac{\partial S_1}{\partial x} = 0\,,$$

with solution

$$\Psi_1 = \tfrac{1}{2}y^2 + \tfrac{1}{24}aAy^4 + y^2 F(x)\,, \quad S_1 = Ay\,. \tag{2.7}$$

The important feature here is the eigenfunction $y^2 F(x)$. It has its counterpart in the Goldstein–Stewartson analysis in the complementary functions $\alpha_i \eta^2$ $(\eta \sim y)$ that appear at the ith stage in their work. The crucial difference is that in their analysis the asymptotic sequence $\{\delta_i(x)\}$ which characterizes the local expansion has to be guessed, subsequent consistency denying or confirming the accuracy of the guess. In the present approach the asymptotic sequence is contained in $F(x)$ and its extraction is more transparent.

Turning to the equations for Ψ_2 and S_2

$$\begin{aligned}&\frac{\partial^2 S_2}{\partial \bar{y}^2} - \bar{y}^2\frac{\partial S_2}{\partial x} = -\frac{A}{k}\,\bar{y}^2 F'(x)\,,\\ &\frac{\partial^3 \Psi_2}{\partial \bar{y}^3} - \bar{y}^2\frac{\partial^2 \Psi_2}{\partial x\,\partial \bar{y}} + 2\bar{y}\frac{\partial \Psi_2}{\partial x} = \frac{aS_2}{k^{\frac{3}{4}}} + \frac{F(x)}{k^{\frac{3}{4}}}\left(\frac{\bar{y}^2}{\sqrt{k}} + \frac{2\bar{y}^2 F}{\sqrt{k}} - \frac{aA\bar{y}^4}{6k}\right),\end{aligned} \tag{2.9}$$

where

$$\bar{y} \equiv k^{\frac{1}{4}} y, \quad k \equiv \tfrac{1}{2}a(B+1) > 0\,,$$

a solution to these equations satisfying the wall conditions can be found for any F. However if we insist that neither Ψ_2 nor S_2 diverge exponentially for large y, then F is uniquely determined. This criterion is precisely the self-consistency check in the Goldstein–Stewartson analysis.

Defining the Laplace Transform

$$\bar{S}_2(p, \bar{y}) = \int_0^\infty e^{-px} S_2(x, \bar{y})\,dx\,,$$

we find

$$\bar{S}_2 = \frac{A\bar{F}'}{pk}\left[1 - \frac{\sqrt{2\bar{y}p^{\frac{1}{4}}}}{\Gamma(\frac{1}{4})} K_{\frac{1}{4}}\left(\tfrac{1}{2}\bar{y}^2 p^{\frac{1}{2}}\right)\right] + \mathscr{F},$$

where $\mathscr{F}$ represents terms that arise only from the initial data and so are regular at separation.

The transformed equation for Ψ_2 may be written

$$\frac{d^3\bar{\Psi}_2}{dY^3} - Y^2\frac{d\bar{\Psi}_2}{dY} + 2Y\bar{\Psi}_2 = Q(Y)\,, \qquad Y = p^{\frac{1}{4}}\bar{y}\,.$$

Then if

$$u = \sqrt{Y}\, I_{\frac{1}{4}}(\tfrac{1}{2}Y^2)\,, \qquad v = \sqrt{Y}\, K_{\frac{1}{4}}(\tfrac{1}{2}Y^2)\,,$$

$\bar{\Psi}_2$ has general solution

$$\frac{d}{dY}\left(\frac{\Psi_2}{Y^3}\right) = \frac{u(Y)}{Y^3}\left[C_1+\tfrac{1}{2}\int^Y V(t)Q(t)t\,dt\right] + \frac{V(Y)}{Y^3}\left[C_2-\tfrac{1}{2}\int^Y u(t)Q(t)t\,dt\right]. \tag{2.11}$$

Then if Ψ_2 is to have a double zero at the wall and not diverge exponentially at infinity,

$$\int_0^\infty tV(t)Q(t)dt = 0 \tag{2.12}$$

and it is this restraint that determines F. In detail,

$$\frac{aA\bar{F}'}{k^{\frac{7}{4}}p^{\frac{1}{2}}}\int_0^\infty t^{\frac{3}{2}}K_{\frac{1}{4}}(\tfrac{1}{2}t^2)\left[1-\frac{\sqrt{2t}}{\Gamma(\frac{1}{4})}K_{\frac{1}{4}}(\tfrac{1}{2}t^2)-\tfrac{1}{6}t^4\right]dt+$$

$$+k^{-\frac{5}{4}}(\bar{F}'+2\bar{F}\bar{F}')\int_0^\infty t^{\frac{7}{2}}K_{\frac{1}{4}}(\tfrac{1}{2}t^2)dt+\mathscr{H}=0 \tag{2.13}$$

where $\mathscr{H}$ arises only from initial data. It is apparent that the initial profiles only add an inhomogeneity to the equation for F, and its precise form is unimportant as far as the nature of F close to separation is concerned. Of crucial importance is the sign of

$$J=\int_0^\infty t^{\frac{3}{2}}K_{\frac{1}{4}}(\tfrac{1}{2}t^2)\left[1-\frac{\sqrt{2t}}{\Gamma(\frac{1}{4})}K_{\frac{1}{4}}(\tfrac{1}{2}t^2)-\tfrac{1}{6}t^4\right]dt$$

which has the value

$$-\sqrt{2}\,\frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}$$

and so is negative. Inverting (2.13) therefore yields

$$-K_1A\int_0^x F'(t)(x-t)^{-\frac{1}{2}}dt+K_2(F'+2FF')+f(x)=0 \tag{2.14}$$

where K_1 and K_2 are both positive constants. With the special choice of initial profiles

$$S_2(0,y)=0,\quad \Psi_2(0,y)\propto y^7$$

and integrating once we find

$$Dx^{\frac{3}{2}}+F+F^2+\tilde{A}\int_0^x F(t)(x-t)^{-\frac{1}{2}}dt=0 \tag{2.15}$$

where $\tilde{A}=-AK_1/K_2$ and it may be noted that

$$K_1=\frac{a}{k^{\frac{7}{4}}}\sqrt{2}\,\frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})}{\pi\Gamma(\frac{1}{4})},\quad K_2=\frac{1}{k^{\frac{5}{4}}}\,\frac{\Gamma(\frac{1}{4})}{\sqrt{2}}.$$

D is a constant.

The essential contribution of the heat transfer is through the integral in (2.15). No matter how small $\tilde{A}$ might be, if it is non-zero it plays a major role in determining the singularity at separation.

3. The Integral Equation

In this section (2.15) is studied in order to deduce some important features of its solution. The domain of interest starts at $x=0$ where F vanishes and ends at the separation point x_s where $F=-\frac{1}{2}$. Since close to the origin

$$F\sim -Dx^{\frac{3}{2}}$$

the discussion will be restricted to the case $D>0$ corresponding to decreasing skin-friction.

If the heat transfer is zero ($\tilde{A}=0$) F has the simple solution

$$F = -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-4\,Dx^{\frac{3}{2}}} \tag{3.1}$$

Separation occurs at

$$x_s = (4D)^{-\frac{2}{3}}$$

and the familiar square-root behavior is rediscovered.

When $\tilde{A}\neq 0$ the discussion depends on whether we are dealing with a hot wall ($\tilde{A}>0$) or a cold wall ($\tilde{A}<0$). Whichever sign is appropriate, when D is positive separation will always occur. To see this for a hot wall assume that separation does not occur so that

$$F > -\tfrac{1}{2} \text{ for all } x$$

It follows that

$$-\tfrac{1}{2} < F^2+F < -Dx^{\frac{3}{2}}+\tilde{A}x^{\frac{1}{2}}$$

which is contradicted for large enough x. When the wall is cold the approach to separation is such that $F''<0$. Certainly F'' is negative in the immediate neighborhood of $x=0$ and we have

$$\tfrac{3}{4}Dx^{-\frac{1}{2}}+F''(1+2F)+2F'^2+\tilde{A}\int_0^x F''(t)(x-t)^{-\frac{1}{2}}\,dt = 0 \tag{3.2}$$

If F'' vanishes at some point all the remaining terms in (3.2) are positive when $\tilde{A}<0$ and we have a contradiction. Integrating F''

$$F < -3Dx^{\frac{3}{2}}$$

and separation must occur for some finite x.

The behavior of F close to the separation point is of fundamental interest so that we define

$$F = -\tfrac{1}{2}+g \qquad g \geqq 0\,,\; g' \leqq 0$$

and then

$$\tfrac{3}{2}Dx^{\frac{1}{2}}+2gg'+\tilde{A}\int_0^x g'(t)(x-t)^{-\frac{1}{2}}\,dt = 0 \tag{3.3}$$

Unless the behavior is pathological, which seems unlikely, gg' tends as $x\to x_s$ to either a constant, zero or infinity. If it approaches a constant this corresponds to a square-root singularity for g and the integral in (3.3) behaves like $\ln(x_s-x)$ as $x\to x_s$. There is nothing to balance this infinite term. If the limit is infinity then there must be a balance between gg' and the integral. This is only possible when $\tilde{A}<0$ (i.e., the wall is cold) and then an expansion can be generated with leading terms

$$g \sim \tfrac{1}{4}\tilde{A}(x_s-x)^{\frac{1}{2}}\ln(x_s-x)+\tfrac{1}{4}\tilde{A}(1-2\ln 2)(x_s-x)^{\frac{1}{2}}\ln\{-\ln(x_s-x)\}+C_3(x_s-x)^{\frac{1}{2}}\ldots \tag{3.4}$$

This cold wall case has been discussed by Buckmaster [2] as a Goldstein–Stewartson expansion and the present work adds no additional information. Numerical integration of (3.3) confirms (3.4) in the sense that the numerical behavior is undistinguishable from the square-root. Nothing more precise can be hoped for since successive terms in the asymptotic sequence decrease very slowly. As $\tilde{A}\to 0$ the first infinity of terms in the expansion (3.4) all vanish except for the third term, and the incompressible limit is retreived.

Turning now to the hot wall case we must have $gg'(x_s)=0$. It follows that the skin-friction vanishes faster than $0(x_s-x)^{\frac{1}{2}}$. Furthermore, numerical integration of (3.3)* reveals that when $\tilde{A}>0$, F vanishes like (x_s-x). It seems that the Stewartson alternative is valid. The expansion does not contain a square-root so that the limit $\tilde{A}\to 0$ is approached somewhat differently than for a cold wall. Numerical evidence is support of the Stewartson alternative for a hot wall is sparse. The calculation of Poots (1960) show no evidence of singular behavior but he was

* A matching process is easily set up with step size decreasing as separation is approached.

not concerned with a highly accurate study of the flow close to separation. Claims have been made (e.g., Brown and Stewartson, [1] that various unpublished work does reveal singular behavior, but in the absence of any published careful examination of this question it would be prudent to reserve judgement. If the Stewartson alternative is found to be valid for a hot wall then the present work gives some mathematical basis for this. If the alternative is incorrect then the present work strongly suggests that resolution of the paradox is unlikely to be found within the framework of the Goldstein–Stewartson analysis. None of our conclusions are modified in character if suction or blowing is permitted at the wall.

It might be wondered whether some different perturbation problem could lead to a singularity. It is difficult to make unequivocal statements but if we insist on being guided by the Goldstein–Stewartson analysis the answer is apparently no. For failure to generate a singularity for a hot wall in the first perturbation Ψ_1, is, in some sense, equivalent to Stewartson's first perturbation f_1, containing no singular contribution to the skin friction. Generation of a singularity at the second perturbation f_2 should be investigated by a study of the perturbation problem

$$\Psi \sim \tfrac{1}{6}a(1+B)y^3+\tfrac{1}{24}\varepsilon aAy^4+\varepsilon^2(y^2/2+y^2F(x))+\varepsilon^3\Psi_3+\dots$$
$$S \sim B+\varepsilon Ay+\varepsilon^3 S_3+\dots$$

where $F(x)$ is determined by requiring Ψ_3 to have an appropriate solution. However, all higher order perturbation problems of this type yield linear equations for F with well-behaved coefficients and there is no mechanism to generate singular behavior.

4. Implications for the Similarity Solution ($m < 0$)

In the Introduction it was mentioned that the similarity solutions of Falkner and Skan behave as m is varied in a manner that seems to reflect the variation of the non-similar boundary-layer with x. Consider the hypothetical duct, an incompressible fluid and $m=-0.09$. We would expect the similarity solution to be appropriate so that the skin-friction is zero. Now if m is decreased a little, the pressure gradient is more adverse and we would expect separation to occur with a consequent breakdown in the boundary layer. It would be difficult to attach any significance to a similarity solution for this smaller m, so that it is gratifying that the solution curve reverses at this point and there are no acceptable solutions for smaller m. Now consider the compressible boundary layer on a hot wall with m chosen so that the skin-friction vanishes. If m is decreased a little, separation will occur, but assuming the conclusions of this paper are valid there is no breakdown of the boundary-layer concept. The separation point is an essential singularity (Stewartson, [9] but plausibly the downstream reversed flow can be adjusted to eliminate this singular behavior and then the physical picture is of a boundary-layer in which the skin-friction vanishes for some finite x and then the flow goes smoothly into a region of reversed flow. There should be a similarity solution to describe the limiting flow with negative skin-friction, and according to the results of Cohen and Reshotko [3], there is.

Continuing the argument, for a cold wall for which breakdown *must* occur when the skin-friction vanishes, we might expect the situation to be no different from the incompressible case. On the contrary, the solution curves of Cohen and Reshotko reverse *before* the skin-friction vanishes. In other words for a given wall temperature there is an m for which a boundary layer exists with positive skin-friction—certainly not an incipient separation situation—but a slight decrease in m cause breakdown. Of course our argument is not rigorous but there is clearly a difficulty here which needs to be understood. Stewartson [10] suggested that separation in the sense of breakdown can no longer be identified with the point of vanishing skin-friction when the flow is compressible, but the evidence is that breakdown can not occur earlier than this for a cold wall, so that this is not the answer. One possibility that suggests itself and is worth speculating about, is connected with the uniqueness of the similarity solutions when $m<0$. It is well known that uniqueness can only be assured if exponential decay is demanded far from the wall. Only one solution has this behavior, all others behaving algebracially. There are no convincing reasons for excluding the algebraic solutions, particularly in the light of the work of Brown and Stewartson [1]. Naturally, the solutions we are interested in are those approached in the limit $x \to \infty$ for a boundary-layer in an inviscid main-stream that is asymptotically $\sim x^m$. Brown and Stewartson [1] claimed that the similarity solutions with exponential decay have an advantage in this respect in that for them the double limit $\lim_{x\to\infty} \lim_{y\to\infty}$ commutes. This is not true however. The boundary layer at finite x decays for large y like

$$y^n A(x) \exp\left[\frac{-(y-k(x))^2}{2F(x)}\right] \tag{4.1}$$

and although the similarity solution also has behavior like this, n is not in general the same in the two cases (k, F and A agree in the limit $x \to \infty$ however). It *is* possible to construct an intermediate solution joining the two, but then it may be possible to construct an intermediate solution joining (4.1) with the algebraically decaying similarity solution. Thus the question is still very much an open one and probably the only way of resolving it is by numerical integration of the full boundary-layer equations. The whole point of this is that if in fact some algebraic behavior *is* appropriate for the similarity solutions then the solution curves of Cohen and Reshotko have to be adjusted and then perhaps the difficulty discussed earlier will disappear.

Acknowledgement

This work was supported by the Office of Naval Research, and also by the Air Force Office of Scientific Research under contract AFOSR. 67–1062.

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